The Lorentz and Poincaré Groups in Relativistic Field Theory

Term Project

Nicolás Fernández González

University of California – Santa Cruz

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Lorentz Group

Our first encounter with the Lorentz group is in special relativity it composed of the transformations that preserve the line element in Minkowski space ($s^2 = \eta_{\mu\nu} x^\mu x^\nu$). In particular, for $x^\mu \to x'^\mu = \Lambda^\mu_\nu x^\nu$, we get:

$$\Lambda^\mu_\rho \eta_{\mu\nu} \Lambda^\nu_\sigma = \eta_{\rho\sigma}$$

Or in matrix notation Λ satisfy the following relation:

$$\mathbf{\Lambda}^\mathsf{T} \eta \mathbf{\Lambda} = \eta \quad \Longrightarrow \quad \mathbf{O}^\mathsf{T} \eta \mathbf{O} = \eta$$

Where η is the matrix:

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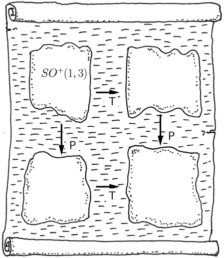
Using our group theory language the Lorentz group is just the indefinite orthogonal group $O(1, 3)$

4 Islands, 2 Boats

The Lorentz group consists of four separated components:

- $\mathcal{L}_+^\uparrow$: det $\Lambda = 1$ and $\Lambda^0_0 \geq 1$.
- $\mathcal{L}_-^\uparrow$: det $\Lambda = -1$ and $\Lambda^0_0 \geq 1$.
- $\mathcal{L}_+^\downarrow$: det $\Lambda = 1$ and $\Lambda^0_0 \leq -1$.
- $\mathcal{L}_-^\downarrow$: det $\Lambda = -1$ and $\Lambda^0_0 \leq -1$.



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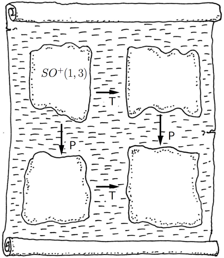

The subgroup of the Lorentz group that exclude spatial reflections and time reversal is called the *proper orthochronous Lorentz group* $SO^+(1,3)$. There are two discrete transformations P and T

$$\begin{aligned}\mathcal{L} &= \mathcal{L}_+^\uparrow \cup \mathcal{L}_-^\uparrow \cup \mathcal{L}_+^\downarrow \cup \mathcal{L}_-^\downarrow \\ &= \mathcal{L}_+^\uparrow \cup P\mathcal{L}_+^\uparrow \cup T\mathcal{L}_+^\uparrow \cup PT\mathcal{L}_+^\uparrow\end{aligned}$$

We get the decomposition of $\mathcal{L}$ into cosets of the restricted group $\mathcal{L}_+^\uparrow$.

The West Coast or East Coast?

Usually we have two possible choices for the Minkowski metric:

- West Coast metric $(1, -1, -1, -1)$. Feynman's favorite
- East Coast metric $(-1, 1, 1, 1)$. The choice of Weinberg and Schwinger (also Ross and Joey?)

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> Checking some books, I remembered one other intriguing recent choice, that of Michael Dine, who wrote the first half of his book (the QFT part) in the West Coast metric, but the second half (the string theory part) in the East Coast metric.
>
> **Update**: For those interested in how to translate back and forth between Coasts in the two-spinor notation, I noticed that Dreiner, Haber and Martin have written review papers, with a line in the tex that lets you choose which Coast. See here and here.

Link: `http://www.math.columbia.edu/~woit/wordpress/`

Generators of the Lorentz group

An infinitesimally Lorentz transformation Λ^μ_ν should be of form,

$$\Lambda^\mu_\nu = \delta^\mu_\nu + \omega^\mu_\nu$$

Where with ω^μ_ν is a matrix of infinitesimal coefficients. Plugging in our definition of Lorentz group and keeping $O(\omega^2)$, we have:

$$\begin{aligned}
\eta_{\rho\sigma} &= \Lambda^\mu_\rho \eta_{\mu\nu} \Lambda^\nu_\sigma \\
\eta_{\rho\sigma} &= \left(\delta^\mu_\rho + \omega^\mu_\rho\right) \eta_{\mu\nu} \left(\delta^\nu_\sigma + \omega^\nu_\sigma\right) \\
\eta_{\rho\sigma} &= \eta_{\rho\sigma} + \omega_{\rho\sigma} + \omega_{\sigma\rho} \\
0 &= \omega_{\rho\sigma} + \omega_{\sigma\rho}
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We conclude that $\omega_{\rho\sigma} = -\omega_{\sigma\rho}$ Thus the generators ω are 4×4 antisymmetric matrices characterized by six parameters so that there are six generators.

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Boost and Rotations

The rotations can be parametrized by a 3-component vector θ_i with $|\theta_i| \leq \pi$, and the boosts by a three component vector β_i (rapidity) with $|\beta| < \infty$. Taking a infinitesimal transformation we have that:

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$$J_1 = i \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & -1 \\ & & 1 & 0 \end{pmatrix} \quad J_2 = i \begin{pmatrix} 0 & & & \\ & 0 & & 1 \\ & & 0 & \\ & -1 & & 0 \end{pmatrix} \quad J_3 = i \begin{pmatrix} 0 & & & \\ & 0 & -1 & \\ & 1 & 0 & \\ & & & 0 \end{pmatrix}$$

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Infinitesimal boost:

$$K_1 = i \begin{pmatrix} 0 & -1 & & \\ 1 & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} \quad K_2 = i \begin{pmatrix} 0 & & -1 & \\ & 0 & & \\ -1 & & 0 & \\ & & & 0 \end{pmatrix} \quad K_3 = i \begin{pmatrix} 0 & & & -1 \\ & 0 & & \\ & & 0 & \\ -1 & & & 0 \end{pmatrix}$$

These matrices are the generators of the Lorentz Group (4-vector basis)

Lie Algebra of the Lorentz Group

Having the generation of any representation we can know the Lie algebra of the Lorentz Group.

$$[J_i, J_j] = i\epsilon_{ijk}J_k$$
$$[J_i, K_j] = i\epsilon_{ijk}K_k$$
$$[K_i, K_j] = -i\epsilon_{ijk}J_k$$

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We can be even more efficient writing as a rank-2 tensor representation

$$
[V_{\mu\nu}, V_{\rho\sigma}] = i\left(V_{\mu\sigma}\eta_{\nu\rho} + V_{\nu\rho}\eta_{\mu\sigma} - V_{\mu\rho}\eta_{\nu\sigma} - V_{\nu\sigma}\eta_{\mu\rho}\right).
$$

Where $V_{\mu\nu}$ is

$$
V_{\mu\nu} = \begin{pmatrix} 0 & K_1 & K_2 & K_3 \\ -K_1 & 0 & J_3 & -J_2 \\ -K_2 & -J_3 & 0 & J_1 \\ -K_3 & J_2 & -J_1 & 0 \end{pmatrix}
$$

So now we can define the Lorentz group as the set of transformations generated by these generators.

We can define a "suspicious" combinations of these two sets of generators as

$$J_i^+ = \frac{1}{2}\left(J_i + iK_i\right) \qquad J_i^- = \frac{1}{2}\left(J_i - iK_i\right)$$

Generators of the Lorentz Group

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$$J_i^+ = \frac{1}{2}(J_i + iK_i) \qquad J_i^- = \frac{1}{2}(J_i - iK_i)$$

Which satisfy the commutation relations,

$$\left[J_i^+, J_j^+\right] = i\epsilon_{ijk}J_k^+$$
$$\left[J_i^-, J_j^-\right] = i\epsilon_{ijk}J_k^-$$
$$\left[J_i^+, J_j^-\right] = 0$$

Black magic! We now see that the algebra of these generators decouple into two $SU(2)$ algebras.

Representations of the Lorentz group

Representations of the Lie algebra of the Lorentz group				
j^+	j^-	Reps	Dim	Field
0	0	$(0,0)$	1	Scalar or Singlet
$\frac{1}{2}$	0	$(\frac{1}{2},0)$	2	Left handed Weyl spinor
0	$\frac{1}{2}$	$(0,\frac{1}{2})$	2	Right handed Weyl spinor
$\frac{1}{2}$	$\frac{1}{2}$	$(\frac{1}{2},\frac{1}{2})$	4	Vector
1	0	$(1,0)$	3	Self-dual 2-form field
0	1	$(0,1)$	3	Anti-self-dual 2-form field
1	1	$(1,1)$	9	Traceless symmetric tensor field.

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- $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$ is a Dirac spinor
- $(1,0)\oplus(0,1)$ is the electromagnetic field tensor $F_{\mu\nu}$

We can label the representations of $so(3,1)$ by the weights (quantum numbers) of $su(2)\otimes su(2)$

We have a problem

Lorentz group are not unitary and that is a deal braking for Physics because we really really like unitary for Quantum Field Theory. Where we were wrong?

Lorentz group are not unitary and that is a deal braking for Physics because we really really like unitary for Quantum Field Theory. Where we were wrong?
It all comes from the factor of i associated with boosts. Or in other words, The precise relationship between the two groups are that the complex linear combinations (complexification) of the generators of the Lorentz algebra to $SU(2) \times SU(2)$.

Poincaré Group

The Poincaré group, as the Lorentz group, preserve the line element in Minkowski space but now with the translations, :
$x^\mu \rightarrow x'^\mu = \Lambda^\mu_\nu x^\nu + a^\mu$

$$(\Lambda_1, a_1) \cdot (\Lambda_2, a_2) = (\Lambda_2 \Lambda_1, \Lambda_2 a_1 + a_2)$$

The Poincaré group has 10 parameters: 6 from a Lorentz and 4 from translations

$$\eta(a, \Lambda) = \begin{pmatrix} & & & & a^0 \\ & & & & a^1 \\ & & \Lambda & & a^2 \\ & & & & a^3 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Poincaré Generators

The Poincaré group is represented by the algebra

$$
\begin{aligned}
[V_{\mu\nu}, V_{\rho\sigma}] &= i\left(V_{\mu\sigma}\eta_{\nu\rho} + V_{\nu\rho}\eta_{\mu\sigma} - V_{\mu\rho}\eta_{\nu\sigma} - V_{\nu\sigma}\eta_{\mu\rho}\right) \\
[P_\mu, P_\nu] &= 0 \\
[V_{\mu\nu}, P_\sigma] &= i\left(P^\mu\eta^{\nu\sigma} - P^\nu\eta^{\mu\sigma}\right)
\end{aligned}
$$

Representations of the Poincaré group

Wigner, on a paper 1939, gave us a method to classify all reducible representations of the Poincaré group by considering the little group of a quantum eigenstate with definite momentum P^2. In the literature is called: Classification of representations by orbits.

P^0	P^2	P^μ	Little Group
m	m^2	$(m,0,0,0)$	$SO(3)$
$-m$	m^2	$(-m,0,0,0)$	$SO(3)$
$\|P\|$	0	$(\|P\|,0,0,\|P\|)$	$E(2)$
$-\|P\|$	0	$(-\|P\|,0,0,\|P\|)$	$E(2)$
0	$-m^2$	$(0,0,0,m)$	$SO(2,1)$
0	0	$(0,0,0,0)$	$SO(3,1)$

Casimir Operators

Finally, there are only two invariants in the Poincaré group that commute with all generators. The first Casimir invariant C_1 is associated with the mass invariance, and the second Casimir invariant C_2 refers to spin invariance.

$$C_1 = P^\mu P_\mu$$
$$C_2 = W_\mu W^\mu$$

Where W_μ is the Pauli-Lubanski (pseudo) vector defined by

$$W_\mu = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} P^\nu V^{\rho\sigma}$$

A particle with fix energy $|\, m, s; P^\mu, \sigma\rangle$